charge effect, we need an *increase* in V of about 0.6% in order to bring the contribution of the isotropic mean-free-path effect to δT_c up to a value $\alpha\rho = 0.93°\text{K}\times 0.060 = 0.05°\text{K}$. (Gayley *et al.*[11] concluded from their data that alloying must increase V.)

Pippard[13] was apparently the first to consider the effect of decreased electron mean free path on V. He stated that this must always *decrease* V for an isotropic metal. However, he considered only longitudinal phonons in his analysis. The transverse phonons also contribute to V, and they would tend to *increase* V when the mean free path decreases.[14] It is possible that this problem will be clarified by further theoretical examination, taking proper account of the contribution of the different phonon modes, and including many-body effects.

In principle, the lattice parameter changes which accompany alloying can also influence ω, N, and V. However, these changes are so small[15] that one is not surprised that they can be ignored.

[13] A. B. Pippard, Phys. Chem. Solids **3**, 175 (1957).
[14] A. B. Pippard, Phil. Mag. **46**, 1104 (1955).
[15] J. A. Lee and G. V. Raynor, Proc. Phys. Soc. (London) **B67**, 737 (1954). For the indium alloys, see the references in Hansen's book (Ref. 4).

Since completion of this work, it has been pointed out to me by Dr. D. Markowitz that his Ph.D. thesis (University of Illinois, 1963, unpublished) contains a prior attempt to analyze the valence effect into a part proportional to $n^{\frac{1}{3}}\delta z$ and a part proportional to ρ, for the tin and aluminum alloys (not for the indium alloys). Our limitation of the discussion to alloys in which the solute is completely dissolved removes several of Dr. Markowitz's apparent exceptions to the hypothesis on which this analysis is based. Our comparison of the results of the analysis with measurements by Glover and Sherrill provides new support for this hypothesis. For a treatment of some of the physical mechanisms which are responsible for the charge effect and the isotropic mean free path effect, the reader is referred to Dr. Markowitz's thesis.

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I am grateful to L. P. Kadanoff for an interesting discussion of the fundamental processes contributing to the valence effect. A preprint sent me by M. H. Cohen, discussing the importance of the electron density in determining the properties of normal metals, helped to stimulate my thinking about the present problem.

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Ultrasonic Attenuation in Superconductors Containing Magnetic Impurities‡

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The Abrikosov-Gor'kov theory for a superconductor with magnetic impurities is applied to the calculation of attenuation coefficients. In this calculation several electronic time-dependent correlation functions are evaluated, and these functions are then used to evaluate the attenuation. In the limit of low sound-wave frequencies, the resulting coefficients are expressed in terms of a single energy integral. The physical significance of the result is discussed in terms of an effective density of "normal electrons" and energy-dependent mean free paths. Then, the attenuation is evaluated in the limit of low temperatures for superconductors both with and without an energy gap.

I. INTRODUCTION

EXPERIMENTS on ultrasonic attenuation have proved to be a most useful way of gaining information about the nature of the superconducting state.[1] Because of experimental difficulties, mostly connected with sample preparation, these studies have hitherto been limited to superconductors which contain a negligible percentage of magnetic impurities. In this paper, we have attacked the problem of calculating the attenuation coefficients in materials containing larger percentages of magnetic impurities in the hope that our work might encourage future experimental work on these materials.

As Abrikosov and Gor'kov[2] first pointed out, superconductors with magnetic impurities are uniquely interesting because they have superconducting properties even in the absence of an energy gap. This behavior

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[1] See, for example, J. R. Leibowitz, Phys. Rev. **133**, A84 (1964); R. Weber, Phys. Rev. **133**, A1487 (1964); E. R. Dobbs and J. M. Perz, Rev. Mod. Phys. **36**, 257 (1964); R. E. Love and R. W. Shaw, Rev. Mod. Phys. **36**, 260 (1964); P. A. Bezuglyi, A. A. Galkin, and A. P. Korolyuk, Zh. Eksperim. i Teor. Phys. (USSR) **39**, 7 (1960) [English transl.: Soviet Phys.—JETP **12**, 4 (1961)].

[2] A. A. Abrikosov and L. P. Gor'kov, Zh. Eksperim. i Teor. Phys. (USSR) **39**, 1781 (1960) [English transl.: Soviet Phys.—JETP **12**, 1243 (1961)].

has been qualitatively verified by Reif and Woolf,[3] although the quantitative fit between theory[2] and this tunneling experiment seems far from perfect.

Our physical model for the superconductor is essentially identical to that of Ref. 2. We assume a weak-coupling electron-gas model for the superconductor very similar to the original model used by Bardeen, Cooper, and Schrieffer.[4] This superconductor has in it randomly placed impurities, some of which are assumed to have a spin attached. The directions of the impurity spin vectors are taken to be random. An interaction proportional to the scalar product of the electronic spin with the impurity spin causes electronic spin flip and thereby seriously modifies the nature of the electronic state. The effect of impurities is described by two parameters Γ_n and Γ_s, which are, respectively, the rate of normal and spin-dependent scattering for a normal-state electron.

We employ this model to calculate several electronic time-dependent correlation functions by a method very similar to the one used by Lange.[5] The correlation functions are then substituted into Tsuneto's[6,7] formulas for the attenuation to obtain explicit results for the attenuation coefficients in the limit of low-frequency sound waves.

Tsuneto's[6] calculation of the ultrasonic attenuation is based upon the determination of response functions of the type

$$\langle[A,B]\rangle(\mathbf{q},z)=\frac{1}{i}\int_{-\infty}^{t}dt'\int d\mathbf{r}'\exp[iz(t-t')-i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')]\times\langle[A(\mathbf{r},t),B(\mathbf{r}',t')]\rangle. \quad (1)$$

Here, $A(\mathbf{r},t)$ and $B(\mathbf{r},t)$ are Heisenberg representation operators describing an observable quantity at the space-time point $\mathbf{r},t$. The $\langle\ \rangle$ denotes both a statistical average and an average over the possible random placements and spin directions of impurities.

The three operators whose correlation functions are needed for ultrasonic attenuation are the number density

$$n(\mathbf{r},t)=\sum_{\text{spin}}\psi^\dagger(\mathbf{r},t)\psi(\mathbf{r},t), \quad (2a)$$

the particle current

$$\mathbf{j}(\mathbf{r},t)=\sum_{\text{spin}}\{[(\nabla-\nabla')/2im][\psi^\dagger(\mathbf{r}',t)\psi(\mathbf{r},t)]\}_{\mathbf{r}'=\mathbf{r}}, \quad (2b)$$

and the electronic stress tensor

$$\tau_{ij}(\mathbf{r},t)=\sum_{\text{spin}}\left\{\frac{(\nabla-\nabla')_i}{2i}\frac{(\nabla-\nabla')_j}{2im}\psi^\dagger(\mathbf{r}',t)\psi(\mathbf{r},t)\right\}_{\mathbf{r}'=\mathbf{r}}. \quad (2c)$$

For waves with wave vector $\mathbf{q}$ pointing in the z direction and angular frequency $v_s q$, the longitudinal attenuation constant is, from the calculation of the Appendix,

$$\alpha_L=-\mathrm{Re}\frac{iz}{\rho_{\text{ion}}v_s}\frac{4\pi e^2}{q^2}\frac{[\langle[n,h_I{}^L]\rangle(\mathbf{q},z)]^2}{1-4\pi e^2q^{-2}\langle[n,n]\rangle(\mathbf{q},z)}-\mathrm{Re}\,iz\langle[h_I{}^L,h_T{}^L]\rangle(\mathbf{q},z)/\rho_{\text{ion}}v_s, \quad (3)$$

where

$$h_I{}^L(\mathbf{r},t)=(q/z)\tau_{zz}(\mathbf{r},t)-(zm/q)n(\mathbf{r},t). \quad (4a)$$

In Eq. (3), z is to be set equal to $v_sq-i\delta$, where δ is an infinitesimal frequency just greater than zero. For all reasonably low frequencies the shielding is almost complete, so that

$$4\pi e^2q^{-2}\langle[n,n]\rangle(\mathbf{q},z)\gg1.$$

For this reason, Eq. (3) may be replaced by

$$\alpha_L=\mathrm{Re}\frac{q^2}{(iz)\rho_{\text{ion}}v_s}\left\{\langle[\tau_{zz},\tau_{zz}]\rangle(\mathbf{q},z)-\frac{[\langle[\tau_{zz},n]\rangle(\mathbf{q},z)]^2}{\langle[n,n]\rangle(\mathbf{q},z)}\right\}. \quad (5a)$$

In an unpublished report,[7] Tsuneto concludes that because the transverse electromagnetic response is very strong in the superconductor

$$\alpha_T=\mathrm{Re}\frac{z^2}{iz\rho_{\text{ion}}v_s}\langle[h_I{}^T,h_I{}^T]\rangle(\mathbf{q},z), \quad (5b)$$

where

$$h_I{}^T(\mathbf{r},t)=(q/z)\tau_{xz}(\mathbf{r},t)-mj_x(\mathbf{r},t). \quad (4b)$$

In our later work, we shall only keep the first term in Eq. (4b), since this is the only term which contributes at low frequencies.

The correlation functions which appear in (3) and (5) are not quite honest electronic correlation functions. Instead, they are correlation functions computed in a fictitious system in which the long-range effects of the electromagnetic interactions are turned off. Therefore, we omit the effects of Maxwell's equations in our calculations of electronic correlation functions because these effects are already included in Eqs. (3) and (5).[8]

[3] F. Reif and M. A. Woolf, Phys. Rev. Letters **9**, 316 (1962).

[4] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).

[5] R. V. Lange, thesis, Harvard University, 1963 (unpublished).

[6] T. Tsuneto, Phys. Rev. **121**, 402 (1961).

[7] Equations (3), (5a), and (5b) are different from those employed by Tsuneto in Ref. 6 in that they include the collision drag effect discussed by R. W. Morse [IBM J. of Res. Develop. **6**, 52 (1962)]; L. T. Claiborne [Ph.D. thesis, Brown University, 1961 (unpublished)]; and J. R. Leibowitz [Phys. Rev. **133**, A84 (1964)]. As these authors all point out, the collision drag effect is dominant in the transverse case for the superconductor. Tsuneto, in a recent unpublished paper, has applied the work of Cohen, Harrison, and Harrison [Phys. Rev. **117**, 937 (1960)] to include collision drag and thereby obtains our Eq. (5b).

[8] The physical source in the denominators of Eqs. (3) and (5a) is the screening of the interaction. The importance of this screening has been emphasized by M. Takimoto, [Progr. Theoret. Phys. (Kyoto) **25**, 327 (1961); **26**, 659 (1961)].

We use a Green's function approach[9–11] to the determination of response functions $\langle[A,B]\rangle(\mathbf{q},z)$. In this approach, it is natural to calculate time-ordered products like $\langle[A(\mathbf{r},t)B(\mathbf{r}',t')]_+\rangle$ for real values of it and it' being between zero and T, where T is the temperature expressed in energy units. The $+$ indicates a time-ordering operation in which operators with larger values of it appear further to the left. The definition of an equilibrium ensemble implies a periodicity condition upon this time-ordered product which can be expressed by writing

$$\langle[A(\mathbf{r},t)B(\mathbf{r}',t')]_+\rangle = -T\int\frac{d^3q}{(2\pi)^3}\sum_\nu \exp[i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')-iz_\nu(t-t')] \times\langle[A,B]\rangle_\nu(\mathbf{q}). \quad (6)$$

In Eq. (6), the sum over ν covers all even integers and $z_\nu = i\pi\nu T$, where T is the temperature in energy units.

In Ref. 11, the relation between the Fourier coefficients of Eq. (6) and the response functions of Eq. (1) are discussed in detail.[12] It turns out that both quantities can be discussed in terms of the spectral weight function $\chi''_{A,B}(\mathbf{q},\omega)$. We have

$$\langle[A,B]\rangle(\mathbf{q},z)=\int\frac{d\omega}{\pi}\frac{\chi''_{A,B}(\mathbf{q},\omega)}{z-\omega}, \quad (7a)$$

$$\langle[A,B]\rangle_\nu(\mathbf{q})=\int\frac{d\omega}{\pi}\frac{\chi''_{A,B}(\mathbf{q},\omega)}{z_\nu-\omega}. \quad (7b)$$

Therefore, it follows immediately that

$$\langle[A,B]\rangle(\mathbf{q},z_\nu)=\langle[A,B]\rangle_\nu(\mathbf{q}). \quad (8)$$

Equation (8) serves to define the Fourier coefficients in terms of the response function $\langle(A,B)\rangle(\mathbf{q},z)$. However, one can use a theorem discussed by Baym and Mermin[13] to travel the opposite route. These authors indicate that the knowledge of $\langle[A,B]\rangle(\mathbf{q},z_\nu)$ for the special points $z_\nu=i\pi\nu T$ is sufficient to determine $\langle[A,B]\rangle(\mathbf{q},z)$ by an analytic continuation procedure when we employ the extra condition that $\langle[A,B]\rangle(\mathbf{q},z)$ has no singularities at $z=\infty$.

Therefore, in this paper our basic calculational procedure is a two-step process. First, we use the Green's function methods to calculate the Fourier coefficients of Eq. (5). Secondly, we use the analytic continuation of these Fourier coefficients and Eq. (8) to find the response function needed in evaluating the ultrasonic attenuation.

In Sec. 2 of this paper, we translate the model we are using into Green's function language by defining an approximation for a Green's function g in the presence of an electromagnetic forcing field. Section 3 includes a brief rederivation of relevant properties of the Green's functions in the absence of the external fields. The solution obtained is identical to that of Abrikosov and Gor'kov, except that we indicate slightly more explicitly the analytic properties of the parameters in the Green's function. In Sec. 4, we expand g to first order in the fields, thereby obtaining solutions for the correlation functions needed in the evaluation of expressions (5). These solutions involve frequency summations, which summations are performed in Sec. 5. Finally, Sec. 6 includes a brief presentation of formulas for the low-frequency case and an explicit evaluation of the low-temperature limit.

II. APPROXIMATION FOR g

In our calculation it is convenient for us to use four-component creation and annihilation operators[14]

$$\Psi(\mathbf{r})=\begin{Bmatrix}\psi_\uparrow(\mathbf{r})\\ \psi_\downarrow(\mathbf{r})\\ \psi_\uparrow^\dagger(\mathbf{r})\\ \psi_\downarrow^\dagger(\mathbf{r})\end{Bmatrix}, \quad (9)$$
$$\Psi^\dagger(\mathbf{r})=(\psi_\uparrow^\dagger(\mathbf{r})\ \psi_\downarrow^\dagger(\mathbf{r})\ \psi_\uparrow(\mathbf{r})\ \psi_\downarrow(\mathbf{r})).$$

We also use the matrices

$$\boldsymbol{\sigma}=\begin{pmatrix}\boldsymbol{\sigma}_p & 0\\ 0 & \boldsymbol{\sigma}_p\end{pmatrix}, \quad (10a)$$

where $\boldsymbol{\sigma}_p$ is the usual set of 2×2 Pauli spin matrices and $\mathbf{0}$ is a 2×2 null matrix and

$$\tau_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \tau_2=i\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\quad \tau_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad (10b)$$

where $\mathbf{1}$ is the two-dimensional unit matrix.

The most basic object in our calculations will be the 4×4 matrix Green's function

$$g(1,1';U,W)=\frac{1}{i}\frac{\langle[S\Psi(1)\Psi^\dagger(1')]_+\rangle}{\langle[S]_+\rangle}, \quad (11)$$

[9] P. C. Martin and J. Schwinger, Phys. Rev. **115**, 1342 (1959).
[10] A. A. Abrikosov, L. Gor'kov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics*, translated by R. A. Silverman (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963).
[11] L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (W. A. Benjamin, Inc., New York, 1963).
[12] See particularly Chap. 8 of Ref. 11.
[13] G. Baym and D. Mermin, J. Math. Phys. **2**, 232 (1961).
[14] These four-component spinor operators have been used by R. Balian. See, for example, Proceedings of the 1963 Ravello Spring School of Physics (Academic Press Inc., to be published).

formed from the spinor creation and annihilation operators. Here $\langle\ \rangle$ stands for both a statistical average and an average over positions and spin directions of the impurities. Following the usual technique,[9,11] the Green's function is defined for pure imaginary times in the interval $0<it<T^{-1}$. The $+$ stands for a Wick time-ordering operation in which the operators are ordered according to the relative size of it. The S has been introduced as a calculational device in order to generate the higher order correlation functions which will be needed in the transport analysis. In particular, we choose

$$S=\exp\left\{-i\int_0^{(iT)^{-1}} d2[U(2)n(2)+W_{ij}(2)\tau_{ij}(2)]\right\}, \quad (12)$$

where n and τ_{ij} are the operators of Eq. (2), while $U(2)$ and $W(2)$ are c number functions which we shall vary at our convenience.

For example,

$$\tfrac{1}{2}\,\mathrm{tr}\left.\frac{\delta g(1,1;U,W)}{\delta U(2)}\right|_{U=W=0} =\langle(n(1)n(2))_+\rangle-\langle n(1)\rangle\langle n(2)\rangle. \quad (13)$$

The term in $\langle n(1)\rangle\langle n(2)\rangle$ is irrelevant in all our calculations of response since it only contributes at precisely zero frequency. The other quantity on the right-hand side of (13) is one of the correlation functions needed in the calculation of α_L.

We now write down the basic approximation used in the determination of $g(1,1';U,W)$. This is precisely the same approximation as employed by previous authors,[2,5] although it is expressed in somewhat different language. We write the matrix equation

$$\int_0^{(iT)^{-1}} d\bar{1}g^{-1}(1,\bar{1};U,W)g(\bar{1},1';U,W)=\delta(1-1'), \quad (14)$$

where

$$g^{-1}(1,1';U,W) =\left\{i\frac{\partial}{\partial t_1}+\left(\frac{\nabla_1{}^2}{2m}+\mu\right)\tau_3-U(1)\tau_3-\frac{\nabla_1-\nabla_1'}{2i}v(1)\right\}\delta(1-1') -\hat{\Delta}(1)\delta(1-1')-\Sigma_n(1,1')-\Sigma_s(1,1'). \quad (15)$$

The first term in Eq. (15) describes the propagation of a free particle through the potential field given by U and v. The next term is

$$\hat{\Delta}(1)=-i|g|\tau_3 g(1,1)\tau_3, \quad (16)$$

which describes the interaction causing the superconductivity (coupling strength $|g|$). The remaining two terms describe the effects of normal and spin-dependent scattering processes, respectively.

If we assume a zero-range interaction between electrons and impurities, then an electron at the point $\mathbf{r}$ sees an interaction potential

$$V(\mathbf{r})=\sum_j [v_n\delta(\mathbf{r}-\mathbf{r}_j)+v_s\delta(\mathbf{r}-\mathbf{r}_j)\mathbf{S}_j\cdot\boldsymbol{\sigma}_p], \quad (17)$$

where v_n and v_s are the strength of the normal and spin-flip interaction while $\mathbf{S}_j$ and $\mathbf{r}_j$ are the spin and position of the jth impurity. This interaction can be written in second quantized form as

$$V=\tfrac{1}{2}\sum_j \Psi^\dagger(\mathbf{r}_j)[v_n\tau_3+v_s\boldsymbol{\alpha}\cdot\mathbf{S}_j]\Psi(\mathbf{r}_j), \quad (18)$$

with

$$\boldsymbol{\alpha}=[(1+\tau_3)/2]\boldsymbol{\sigma}+[(1-\tau_3)/2]\sigma_2\boldsymbol{\sigma}\sigma_2. \quad (19)$$

These normal and spin-flip scatterings lead to the Σ_n and Σ_s of Eq. (15). In lowest order,[15]

$$\begin{aligned}\Sigma_n(1,1')&=(\pi\Gamma_n/mp_F)\delta(\mathbf{r}_1-\mathbf{r}_1')\tau_3 g(1,1')\tau_3,\\ \Sigma_s(1,1')&=(\pi\Gamma_s/mp_F)\delta(\mathbf{r}_1-\mathbf{r}_1')\tfrac{1}{3}\boldsymbol{\alpha}\cdot g(1,1')\boldsymbol{\alpha}.\end{aligned} \quad (20)$$

Here, Γ_n and Γ_s are the rates for normal and spin-flip scattering for an electron at the surface of the Fermi sphere in the normal state:

$$\begin{aligned}\Gamma_n&=[|v_n|^2 mp_F/\pi]n_I,\\ \Gamma_s&=[|v_s|^2 mp_F/\pi]n_I\langle S_j(S_j+1)\rangle,\end{aligned} \quad (21)$$

where p_F is the Fermi momentum, m the electronic mass, n_I the impurity density, and $\langle S_j(S_j+1)\rangle$ the average of $\mathbf{S}_j{}^2$.

III. EQUILIBRIUM GREEN'S FUNCTION

For the case $U=W=0$, the solution to Eq. (14) is well-known[2,5] so we shall only very briefly present results here. In this case,

$$g(1,1')=iT\sum_\nu\int\frac{d^3p}{(2\pi)^3}g_\nu(\mathbf{p}) \times\exp[i\mathbf{p}(\mathbf{r}_1-\mathbf{r}_1')-iz_\nu(t_1-t_1')], \quad (22)$$

where, as before, $z_\nu=i\pi\nu T$, but now the sum is over all odd integers. Then, Eq. (14) may be written as

$$[g_\nu(\mathbf{p})]^{-1}=z_\nu-\epsilon_p\tau_3-\hat{\Delta}-\frac{\Gamma_n}{2\pi}\tau_3\int d\epsilon_p g_\nu(\mathbf{p})\tau_3 -\frac{\Gamma_s}{2\pi}\frac{1}{3}\boldsymbol{\alpha}\cdot\int d\epsilon_p g_\nu(\mathbf{p})\tau_3\boldsymbol{\alpha}, \quad (23)$$

where

$$\epsilon_p=p^2/2m-\mu. \quad (24)$$

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[15] The derivation of these terms can be achieved by a generalization of the type of analysis used in Sec. 39 of Ref. 10 or in Chap. 1, Sec. 4 of Ref. 5.

We have used the fact that only particles with momenta near the Fermi surface contribute to write

$$\int \frac{d^3p}{(2\pi)^3} = \frac{mp_F}{2\pi^2} \int \frac{d\Omega}{4\pi} \int d\epsilon_p, \tag{25}$$

with $d\Omega$ being an element of solid angle.

Next, we must analytically continue $g_\nu(\mathbf{p})$ to the function $g(\mathbf{p},z)$. The latter must satisfy $g(\mathbf{p},z_\nu)=g_\nu(\mathbf{p})$. The obvious way of meeting this requirement is to demand that $g(p,z)$ satisfy Eq. (14), i.e., that

$$[g(\mathbf{p},z)]^{-1} = z - \epsilon_p\tau_3 - \hat{\Delta} - \frac{\Gamma_n}{2\pi}\tau_3 \int d\epsilon_p g(\mathbf{p},z)\tau_3 - \frac{\Gamma_s}{2\pi}\frac{\boldsymbol{\alpha}}{3}\cdot \int d\epsilon_p g(\mathbf{p},z)\boldsymbol{\alpha}. \tag{26}$$

To solve Eq. (26), we assume that

$$[g^{-1}(\mathbf{p},z)]^{-1} = \tilde{\omega}(z) - \epsilon_p\tau_3 - \tilde{\Delta}(z)\tau_1\sigma_1. \tag{27}$$

Then, $\hat{\Delta}$ has the form

$$\hat{\Delta} = \Delta\tau_1\sigma_1, \tag{28}$$

where Δ is real. The integral in (26) can be performed to give

$$\int d\epsilon g(\mathbf{p},z) = -\pi i \frac{\tilde{\omega}(z)+\tilde{\Delta}(z)\tau_1\sigma_1}{\tilde{\epsilon}(z)}, \tag{29}$$

where $\tilde{\epsilon}(z)$ is the square root defined by

$$\tilde{\epsilon}(z) = \{[\tilde{\omega}(z)]^2 - [\tilde{\Delta}(z)]^2\}^{1/2} \tag{30}$$

and the condition

$$\text{Im}\tilde{\epsilon}(z) > 0. \tag{31}$$

Equations (26), (27), and (29) now define $\tilde{\omega}(z)$ and $\tilde{\Delta}(z)$ by the equations

$$\begin{aligned} \tilde{\omega}(z) &= z + i\frac{\Gamma_n+\Gamma_s}{2}\frac{\tilde{\omega}(z)}{\tilde{\epsilon}(z)}, \\ \tilde{\Delta}(z) &= \Delta + i\frac{\Gamma_n-\Gamma_s}{2}\frac{\tilde{\Delta}(z)}{\tilde{\epsilon}(z)}. \end{aligned} \tag{32}$$

We do not use $\tilde{\omega}$ and $\tilde{\Delta}$ as our basic functions. Instead, we employ $\tilde{\epsilon}(z)$ and

$$u(z) = \tilde{\omega}(z)/\tilde{\Delta}(z). \tag{33}$$

From Eq. (21), it follows that

$$\tilde{\omega}(z) = \tilde{\epsilon}(z)\{u(z)/[[u(z)]^2-1]^{1/2}\}, \tag{34a}$$

$$\tilde{\Delta}(z) = \tilde{\epsilon}(z)/\{[u(z)]^2-1\}^{1/2}, \tag{34b}$$

while Eq. (32) implies that $u(z)$ and $\tilde{\epsilon}(z)$ satisfy

$$\begin{aligned} \tilde{\epsilon}(z) &= \tfrac{1}{2}i(\Gamma_n+\Gamma_s) + z\{[u(z)]^2-1\}^{1/2}/u(z) \\ &= \tfrac{1}{2}i(\Gamma_n-\Gamma_s) + \Delta\{[u(z)]^2-1\}^{1/2}, \end{aligned} \tag{35}$$

$$z/\Delta = u(z)\{1-(i\Gamma_s/\Delta)[[u(z)]^2-1]^{-1/2}\}. \tag{36}$$

We shall need to know the analytic structure of $u(z)$, $\tilde{\epsilon}(z)$, of $\tilde{\omega}(z)$, $\tilde{\Delta}(z)$ for our further analysis. The structure of these functions is derivable from the analytic structure of $g(p,z)$, which is as follows:

(i) $g(\mathbf{p},z)$ is an analytic function of z except for a cut along the real axis.

(ii) The discontinuity across the cut comes totally from the fact that $\text{Im}g(\mathbf{p},z)$ changes sign as z crosses the real axis.

(iii) The imaginary part of the usual Green's function (not the matrix function) is never positive for z lying just above the real axis. Thus

$$\text{Im tr}[(1+\tau_3)/4]g(\mathbf{p},z)\,|_{z=\omega+i\delta} \leq 0$$

or

$$\text{Im}\frac{\tilde{\omega}(z)+\epsilon}{[\tilde{\omega}(z)]^2-\epsilon^2-[\tilde{\Delta}(z)]^2} \leq 0. \tag{37}$$

Using these facts, we can conclude that

(i) $\tilde{\Delta}(z)$, $\tilde{\omega}(z)$, $\tilde{\epsilon}(z)$, and $u(z)$ are also analytic except for cuts along the real axis. We call the values of these functions when z lies just above the real axis, $z=\omega+i\delta$, $\tilde{\Delta}(\omega)$, $\tilde{\omega}(\omega)$, etc.

(ii) For $\tilde{\Delta}$, $\tilde{\omega}$, and hence u the discontinuity across the real axis is just given by the imaginary part of the function, i.e.,

$$\begin{aligned} \tilde{\Delta}(\omega-i\delta) &= \tilde{\Delta}^*(\omega), \\ \tilde{\omega}(\omega-i\delta) &= \tilde{\omega}^*(\omega), \\ u(\omega-i\delta) &= u^*(\omega), \\ \tilde{\epsilon}(\omega-i\delta) &= -\tilde{\epsilon}^*(\omega), \\ \{[u(\omega-i\delta)]^2-1\}^{1/2} &= -\{[[u(\omega)]^2-1]^{1/2}\}^*. \end{aligned} \tag{38}$$

The first three lines of Eq. (38) are directly derivable from the fact that $g(\mathbf{p},\omega)$ obeys a relationship analogous to (38). The two last lines follow from Eqs. (32) and (36).

(iii) By taking the ϵ integral of Eq. (37) and employing Eq. (29), we find

$$-\text{Im}\tfrac{1}{2}\,\text{tr}(1+\tau_3)\int d\epsilon g(\mathbf{p},z)\Big|_{z=\omega+i\delta} = \text{Re}\frac{u(\omega)}{\{[u(\omega)]^2-1\}^{1/2}} > 0. \tag{39}$$

Then Eq. (36) implies

$$\text{Im}u(\omega) > 0. \tag{40}$$

IV. SOLUTION FOR $\mathcal{L}$

In the previous section, we calculated $g(1,1';U,W)$ for $U=W=0$. For our ultrasonic attenuation calculations, we need the first order terms in the expansion of g in a power series in U and V, i.e.,

$$\mathcal{L}_n(1,2;1',2)=\frac{\delta g(1,1';U,v)}{\delta U(2)}\bigg|_{U=W=0}=-T^2\sum_{\nu_+,\nu_-}\int\frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\mathcal{L}_n(\mathbf{p},\mathbf{q},z_+,z_-)\times\exp\{i\mathbf{p}\cdot(\mathbf{r}_1-\mathbf{r}_1')+i\mathbf{q}\cdot[\tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_1')-\mathbf{r}_2]-iz_+(t_1-t_2)+iz_-(t_1'-t_2)\}\,. \quad (41)$$

In Eq. (41), $z_\pm=i\pi\nu_\pm T$ and the sum over ν_+ and ν_- ranges over all odd integers. The variation with respect to $W_{ij}(2)$ yields

$$\mathcal{L}_{\tau ij}(1,2;1',2)=[\delta g(1,1')/\delta W_{ij}(2)]_{U=W=0}\,. \quad (42)$$

We define $\mathcal{L}_\tau(\mathbf{p},\mathbf{q},z_+,z_-)$ in a manner directly analogous to the definition of Eq. (41). These two $\mathcal{L}$'s are sufficient for calculating all the correlation functions needed for transport since, for example,

$$\langle[n,n]\rangle(\mathbf{q},z_\nu)=\frac{T}{2}\int\frac{d^3p}{(2\pi)^3}\sum_{\nu+}\mathrm{tr}[\tau_3\mathcal{L}_n(\mathbf{p},\mathbf{q},z_+,z_-)]\quad z_+=i\pi\nu_+T,\quad z_-=i\pi(\nu_+-\nu)T\,. \quad (43)$$

For notational convenience, we define the quantities $\langle[A,B]\rangle(\mathbf{q},z_+,z_-)$, which have the property that

$$\langle[A,B]\rangle(\mathbf{q},z_\nu)=T\sum_{\nu+}\langle[A,B]\rangle(\mathbf{q},i\pi(\nu_++\nu)T,i\pi\nu_+T)\,. \quad (44)$$

Thus, we have

$$\langle[n,n]\rangle(\mathbf{q},z_+,z_-)=\int\frac{d^3p}{(2\pi)^3}\tfrac{1}{2}\,\mathrm{tr}\tau_3\mathcal{L}_n(\mathbf{p},\mathbf{q},z_+,z_-)\,, \quad (45a)$$

$$\langle[\tau_{ij},n]\rangle(\mathbf{q},z_+,z_-)=\int\frac{d^3p}{(2\pi)^3}\tfrac{1}{2}\,\mathrm{tr}\tau_3\frac{p_ip_j}{m}\mathcal{L}_n(\mathbf{p},\mathbf{q},z_+,z_-)=\int\frac{d^3p}{(2\pi)^3}\tfrac{1}{2}\,\mathrm{tr}\tau_3\mathcal{L}_{\tau ij}(\mathbf{p},\mathbf{q},z_+,z_-)\,, \quad (45b)$$

$$\langle[\tau_{ij},\tau_{kl}]\rangle(q,z_+,z_-)=\int\frac{d^3p}{(2\pi)^3}\tfrac{1}{2}\,\mathrm{tr}\tau_3\frac{p_ip_j}{m}\mathcal{L}_{\tau kl}(\mathbf{p},\mathbf{q},z_+,z_-)\,. \quad (45c)$$

To calculate the $\mathcal{L}$'s we notice that[16]

$$\begin{aligned}\mathcal{L}_n(1,2;1',2)&=\delta g(1,1')/\delta U(2)\\&=-\int_0^{iT^{-1}}d\bar{1}d\bar{1}'g(1,\bar{1})\frac{\delta g^{-1}(\bar{1},\bar{1}')}{\delta U(2)}g(\bar{1}',1')\\&=g(1,2)\tau_3g(2,1)+\int d\bar{1}g(1,\bar{1})\frac{\delta\hat{\Delta}(\bar{1})}{\delta U(2)}g(1,1')\\&\quad+\frac{\pi\Gamma_n}{mp_F}\int d\bar{1}d\bar{1}'g(1,\bar{1})\delta(\bar{r}_1-\bar{r}_1')\tau_3\frac{\delta g(\bar{1},\bar{1}')}{\delta U(2)}\tau_3g(1',1')\\&\quad+\frac{\pi\Gamma_s}{mp_F}\int d\bar{1}d\bar{1}'g(1,\bar{1})\delta(\bar{r}_1-\bar{r}_1')\frac{\boldsymbol{\alpha}}{3}\cdot\frac{\delta g(\bar{1},\bar{1}')}{\delta U(2)}\times\boldsymbol{\alpha}g(\bar{1}',1')\,. \quad (46)\end{aligned}$$

The equation for $\mathcal{L}_{\tau ij}$ is identical in form except that for $\mathcal{L}_{\tau ij}$ the inhomogeneous term is

$$\left[\left(\frac{\nabla_2-\nabla_2'}{2i}\right)_i\left(\frac{\nabla_2-\nabla_2'}{2im}\right)_jg(1,2')\tau_3g(2,1')\right]_{2'=2}\,. \quad (47)$$

The term in $\delta\hat{\Delta}(1)/\delta U(2)$ in Eq. (46) reflects the presence of a collective mode like the one originally discussed by Anderson. Tsuneto[6] has shown that this collective mode does not affect the ultrasonic attenuation for the case of ordinary impurity scattering. Consequently, we neglect $\delta\hat{\Delta}(1)/\delta U(2)$ in Eq. (46).[17]

The Fourier transform of Eq. (46) is introduced with the aid of Eq. (41). Thus,

$$\mathcal{L}_n(\mathbf{p},\mathbf{q},z_+,z_-)=g(\mathbf{p}+\mathbf{q}/2,z_+)\{\ \}g(\mathbf{p}-\mathbf{q}/2,z_-)\,,$$

$$\{\ \}=\tau_3+\frac{\pi\Gamma_n}{mp_F}\int\frac{d^3p'}{(2\pi)^3}\tau_3\mathcal{L}_n(\mathbf{p}',\mathbf{q},z_+,z_-)\tau_3+\frac{\pi\Gamma_s}{3mp_F}\int\frac{d^3p'}{(2\pi)^3}\boldsymbol{\alpha}\cdot\mathcal{L}_n(\mathbf{p}',\mathbf{q},z_+,z_-)\boldsymbol{\alpha}\,. \quad (48)$$

In the transverse case, in which we wish to calculate $\mathcal{L}_{\tau xz}$, the analogs of the final two terms in Eq. (48) do not contribute since it is impossible to form a scalar from $\mathcal{L}_{\tau xz}$ and a wave vector in the z direction. Thus, the transverse correlation function can be obtained immediately by substituting expression (47) into Eq. (45c) and then performing the momentum integral.

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[16] Y. Osaka, in J. Phys. Soc. Japan **18**, 877 (1963), has stressed the importance of constructing approximations of $\mathcal{L}_n$ and $\mathcal{L}_\tau$ that satisfy gauge invariance, i.e., the differential number conservation law. Baym and Kadanoff [Phys. Rev. **124**, 287 (1961)] have indicated why approximations like (46) do, in fact, include all conservation laws.

[17] When this term is dropped, the solutions for $\mathcal{L}$ are no longer gauge invariant.

The momentum integrals needed to calculate the response functions defined by Eq. (45) can be evaluated directly, if we assume $q^2/2m\ll\mu$, so that

$$\int\frac{d^3p}{(2\pi)^3}\approx\frac{mp_F}{(2\pi)^3}\int d\epsilon d\Omega\,,$$

$$\epsilon_{\mathbf{p}+\mathbf{q}/2}=\epsilon_p+\tfrac{1}{2}v_Fq\cos\theta\,, \qquad (49)$$

where θ is the angle between $\mathbf{p}$ and $\mathbf{q}$. Thus, we find

$$\begin{aligned}\langle[\tau_{xz},\tau_{xz}]\rangle(\mathbf{q},z_+,z_-)\\ &=\tfrac{1}{2}\,\mathrm{tr}\tau_3\int\frac{d^3p}{(2\pi)^3}\frac{p_x^2p_z^2}{m^2}g(\mathbf{p}+\mathbf{q}/2,\,z_+)\tau_3g(\mathbf{p}-\mathbf{q}/2,\,z_-)\\ &=\frac{p_F^4}{2\pi q}[1-c(z_+,z_-)]Y^{-4}[\tfrac{2}{3}Y^3+Y-(Y^2+1)\tan^{-1}Y]\\ &=\frac{p_F^4}{3\pi q}[1-c(z_+,z_-)]Y^{-1}[1-g(Y)]\,, \qquad (50)\end{aligned}$$

where

$$\begin{aligned}C(z_+,z_-)&=\frac{u(z_+)u(z_-)-1}{[u(z_+)^2-1]^{1/2}[u(z_-)^2-1]^{1/2}}\\ &=\frac{\tilde\omega(z_+)\tilde\omega(z_-)-\tilde\Delta(z_+)\tilde\Delta(z_-)}{\tilde\epsilon(z_+)\tilde\epsilon(z_-)}\,, \qquad (51a)\end{aligned}$$

and

$$Y=Y(z_+,z_-,\mathbf{q})=iqv_F/[\tilde\epsilon(z_+)+\tilde\epsilon(z_-)]\,, \qquad (51b)$$

and

$$g(Y)=\tfrac{3}{2}Y^{-3}\{-Y+(Y^2+1)\tan^{-1}Y\}\,. \qquad (52)$$

The longitudinal correlation functions are a bit harder to evaluate because Eq. (48) is an integral equation. To solve this equation, we integrate Eq. (48) over all p and try a solution for $\mathfrak{L}_n$ of the form

$$\int\frac{d^3p}{(2\pi)^3}\mathfrak{L}_n(\mathbf{p},\mathbf{q},z_+,z_-) =\tfrac{1}{2}\tau_3\langle[n,n]\rangle(\mathbf{q},z_+,z_-)+\tfrac{1}{2}\tau_2\sigma_1f(\mathbf{q},z_+,z_-)$$

and a solution of a similar form for $\int d^3p\mathfrak{L}_{\tau_{zz}}(\mathbf{p},\mathbf{q},z_+,z_-)$. After considerable algebra, the correlation functions emerge as

$$\langle[n,n]\rangle(\mathbf{q},z_+,z_-) =\frac{m^2}{\pi q}[1-c(z_+,z_-)]\frac{X\tan^{-1}Y}{X-\tan^{-1}Y}\,, \qquad (53)$$

$$\langle[n,\tau_{zz}]\rangle(\mathbf{q},z_+,z_-) =\frac{p_F^2m}{\pi q}[1-c(z_+,z_-)]XY^{-2}\frac{Y-\tan^{-1}Y}{X-\tan^{-1}Y}\,, \qquad (54)$$

$$\langle[\tau_{zz},\tau_{zz}]\rangle(\mathbf{q},z_+,z_-) =\frac{p_F^2}{\pi q}[1-c(z_+,z_-)]Y^{-4}\left[\tfrac{1}{3}Y^3+\frac{Y-\tan^{-1}Y}{X-\tan^{-1}Y}(Y-X)\right]\,, \qquad (55)$$

with

$$X=X(\mathbf{q},z_+,z_-)=qv_F/[\Gamma_n-c(z_+,z_-)\Gamma_s]\,. \qquad (56)$$

V. FREQUENCY SUMS

To form the needed correlation functions, $\langle[A,B]\rangle\times(\mathbf{q},z)$, it is necessary to perform frequency sums on $\langle[A,B]\rangle(\mathbf{q},z_+,z_-)$. According to Eq. (44)

$$\langle[A,B]\rangle(\mathbf{q},i\pi\nu T) =T\sum_{\nu_+}\langle[A,B]\rangle(\mathbf{q},i\pi(\nu_++\nu)T,i\pi\nu_+T)\rangle\,, \qquad (57)$$

where the sum over ν_+ covers all odd integers and ν is an even integer. This sum may be evaluated by noticing that all the singularities of $\langle[A,B]\rangle(\mathbf{q},z_+,z_-)$ appear for real values of z_+ and z_-. Hence, this function has a spectral representation

$$\langle[A,B]\rangle(\mathbf{q},z_+,z_-)=\int\frac{d\omega_+}{2\pi}\frac{d\omega_-}{2\pi}\frac{\chi_{A,B}(\mathbf{q},\omega_+,\omega_-)}{(z_+-\omega_+)(z_--\omega_-)}\,. \qquad (58)$$

Here, χ is the real quantity

$$\begin{aligned}\chi_{A,B}(\mathbf{q},\omega_+,\omega_-)=&-\langle[A,B]\rangle(\mathbf{q},\,\omega_++i\delta,\,\omega_-+i\delta)\\ &-\langle[A,B]\rangle(\mathbf{q},\,\omega_+-i\delta,\,\omega_--i\delta)\\ &+\langle[A,B]\rangle(\mathbf{q},\,\omega_++i\delta,\,\omega_--i\delta)\\ &+\langle[A,B]\rangle(\mathbf{q},\,\omega_+-i\delta,\,\omega_-+i\delta)\,. \qquad (59)\end{aligned}$$

Equation (58) may be substituted into Eq. (57) and the summation performed with the aid of the Summerfeld-Watson sum formula. The result is

$$\langle[A,B]\rangle(\mathbf{q},i\pi\nu T) =\int\frac{d\omega_+}{2\pi}\frac{d\omega_-}{2\pi}[f(\omega_+)-f(\omega_-)]\frac{\chi_{A,B}(\mathbf{q},\omega_+,\omega_-)}{\omega_+-i\pi\nu T-\omega_-}\,, \qquad (60a)$$

with $f(\omega)=[1+e^{\omega/T}]^{-1}$. The analytic continuation of Eq. (60a) is simple and obvious: We need to construct a function $\langle[A,B]\rangle(\mathbf{q},z)$ which agrees with the right-hand side of (60a) at all the special points $z=z_\nu=i\pi\nu T$ for even integral ν. This function should have no singularity at $z=\infty$. Clearly a possible choice of such a function is

$$\langle[A,B]\rangle(\mathbf{q},z) \times\int\frac{d\omega_+}{2\pi}\frac{d\omega_-}{2\pi}[f(\omega_+)-f(\omega_-)]\frac{\chi_{A,B}(\mathbf{q},\omega_+,\omega_-)}{\omega_+-z-\omega_-}\,. \qquad (60b)$$

According to Baym and Mermin,[18] it is the only such function.

The right-hand side of Eq. (60b) can be rewritten in terms of the known functions $\langle[A,B]\rangle(\mathbf{q},z_+,z_-)$ by making use of the definition (59) of χ in terms of these functions. This gives

$$\langle[A,B]\rangle(\mathbf{q},z) = \int \frac{d\omega}{2\pi i} f(\omega)\{\langle[A,B]\rangle(\mathbf{q},\omega-i\delta,\omega-z) - \langle[A,B]\rangle(\mathbf{q},\omega+i\delta,\omega-z) + \langle[A,B]\rangle(\mathbf{q},\omega+z,\omega-i\delta) - \langle[A,B]\rangle(\mathbf{q},\omega+z,\omega+i\delta)\}. \quad (61)$$

Equation (61) should permit the calculation of all the response functions needed for the attenuation coefficients.

Unfortunately, Eq. (61) is wrong. It is wrong because of a curious conditional convergence[18] of the sum and integral in $\int d\epsilon \sum_\nu \mathcal{L}(\mathbf{p},\mathbf{q},z_+,z_-)$. In the interchange of orders of integration and summation, we get an extra spurious term which comes from very large z_+ and ϵ. Since this term comes from high energies, it is almost independent of z—it does not vary appreciably throughout the entire range $|z|\ll\mu$. Another consequence of the high-energy source of this term is that it can be estimated perfectly well from the normal-state Green's functions.

Thus, we write instead of (60b)

$$\langle[A,B]\rangle(\mathbf{q},z) = \langle[A,B]\rangle_\infty + \int \frac{d\omega_+}{2\pi}\frac{d\omega_-}{2\pi}\frac{\chi_{A,B}(\mathbf{q},\omega_+,\omega_-)}{z_- - \omega_+ + \omega_-}[f(\omega_+)-f(\omega_-)], \quad (62)$$

where

$$\begin{aligned}\langle[n,n]\rangle_\infty &= -mp_F/\pi^2,\\ \langle[n,\tau_{zz}]\rangle_\infty &= -N,\\ \langle[\tau_{zz},\tau_{zz}]\rangle_\infty &= -\tfrac{3}{5}(Np_F^2/m),\\ \langle[\tau_{xz},\tau_{xz}]\rangle_\infty &= -\tfrac{1}{5}Np_F^2/m,\end{aligned} \quad (63)$$

are the high-frequency contributions to their respective response. These terms should be added on to the right-hand side of Eq. (61) in order to make that equation correct.

VI. FORMULAS FOR LOW-FREQUENCY ATTENUATION COEFFICIENTS

Equation (61) can, in principle, be used to study attenuation for all frequencies. The algebraic complexity is so fearsome, however, that we shall limit ourselves to the case of relatively low-frequency sound. Therefore, we expand Eq. (61) about $z=0$. We can simplify this expansion by noting that, for all our cases, $\langle[A,B]\rangle \times(q,z_+,z_-)$ is even under the interchange of z_+ and z_-. The low-frequency limit is

$$\langle[A,B]\rangle(\mathbf{q},z) = \langle[A,B]\rangle_\infty - \int \frac{d\omega}{\pi} f(\omega)\,\mathrm{Im}\langle[A,B]\rangle(\mathbf{q},\omega+i\delta,\omega+i\delta) + iz\int\frac{d\omega}{2\pi}\frac{\partial f(\omega)}{\partial\omega}\{\mathrm{Re}\langle[A,B]\rangle(\mathbf{q},\omega+i\delta,\omega+i\delta) - \langle[A,B]\rangle(\mathbf{q},\omega+i\delta,\omega-i\delta)\}. \quad (64)$$

Therefore, Eq. (64) may be simplified to read

$$\langle[A,B]\rangle(\mathbf{q},z) = \langle[A,B]\rangle_\infty - iz\int\frac{d\omega}{2\pi}\frac{\partial f(\omega)}{\partial\omega} \times\langle[A,B]\rangle(\mathbf{q},\omega+i\delta,\omega-i\delta). \quad (65)$$

By employing Eq. (5b), (50), (52), and (65) we can now directly read off

$$\alpha_T = -\frac{qNmv_F}{p_{\mathrm{ion}}v_s}\frac{1}{2}\int d\omega\frac{\partial f(\omega)}{\partial\omega}[1-c(\omega)]Y^{-1} \times[1-g(y)], \quad (66)$$

where

$$c(\omega)=c(\omega-i\delta,\omega+i\delta) = -[|u(\omega)|^2-1]/|[u(\omega)]^2-1|, \quad (67)$$

$$y=y(\omega)=qv_F/[2\,\mathrm{Im}\,\tilde{\epsilon}(\omega)] = qv_F/\{2\,\mathrm{Im}[[\tilde{\omega}(\omega)]^2-[\tilde{\Delta}(\omega)]^2]^{1/2}\}. \quad (68)$$

The longitudinal case is a bit more complex. From Eqs. (5a), (63), and (65), we find that the low-frequency attenuation takes the form

$$\alpha_L = \mathrm{Re}\frac{q^2}{izp_{\mathrm{ion}}v_s}\left\{\langle[\tau_{zz},\tau_{zz}]\rangle(\mathbf{q},z) - 2\frac{p_F^2}{3m}\langle[\tau_{zz},n]\rangle(\mathbf{q},z) + \left(\frac{p_F^2}{3m}\right)^2\langle[n,n]\rangle(\mathbf{q},z)\right\}. \quad (69)$$

Equation (65) may be employed together with Eqs. (53), (54), and (55) to evaluate (69) as

$$\alpha_L = -\frac{qNmv_F}{p_{\mathrm{ion}}v_s}\frac{1}{2}\int d\omega\frac{\partial f(\omega)}{\partial\omega}[1-c(\omega)] \times\frac{x(\omega)+3y^{-2}[x(\omega)-y]}{x(\omega)-\tan^{-1}y} \times[\tfrac{1}{3}\tan^{-1}y - y^{-2}(y-\tan^{-1}y)], \quad (70)$$

with

$$x(\omega)=X(\mathbf{q},\omega-i\delta,\omega+i\delta) = qv_F/[\Gamma_n-\Gamma_s c(\omega)]. \quad (71)$$

[18] This behavior is noted by Tsuneto (Ref. 6) and discussed in detail in Ref. 10, p. 310.

For an elementary check of this formula, consider the normal state for which

$$c(\omega)=-1\,,$$
$$y(\omega)=x(\omega)=qv_F/(\Gamma_n+\Gamma_s)=ql\,; \tag{72}$$

and thus

$$\alpha_L=\alpha_L{}^N=\frac{mNv_F}{\rho_{\text{ion}}v_s l}\times\left\{\frac{1}{3}\frac{(ql)^2\tan^{-1}(ql)}{(ql)-\tan^{-1}(ql)}-1\right\}, \tag{73}$$

which is Pippard's[19] formula for the longitudinal attenuation constant.

This comparison enables us to see a relatively simple interpretation of Eqs. (66) and (70). We can interpret $y(\omega)$ as q times an effective frequency-dependent mean free path. Then Eq. (66) is identical to Pippard's result for the longitudinal case, except for the factor $[1-c(\omega)]$, which is just a typical superconducting coherence factor. Our results in the transverse case are essentially different from Pippard's because the transverse electromagnetic response is qualitatively larger in the superconductor. In the transverse case, the normal attenuation constant as given by Pippard[19] is

$$\alpha_T{}^N=\frac{qNmv_F}{\rho_{\text{ion}}v_s}\left[\frac{1-g(ql)}{g(ql)}\right]\frac{1}{ql}, \tag{74}$$

where $g(ql)$ is given by Eq. (52).

There are several relatively tractable cases in which Eqs. (66) and (70) may be evaluated analytically. For example, when $\Gamma_n\gg\Gamma_s$ and $\Gamma_n\gg\Delta$, $x(\omega)=y(\omega)=ql$. In this situation,[20]

$$\alpha_L/\alpha_L{}^N=(\rho^N/\rho)\,, \tag{75}$$

$$\alpha_T/\alpha_T{}^N=(\rho^N/\rho)g(ql)\,, \tag{76}$$

where

$$\rho^N/\rho=-\frac{1}{2}\int d\omega\frac{\partial f(\omega)}{\partial\omega}[1-c(\omega)] \tag{77}$$

can be viewed as the proportion of "normal electrons" effectively acting in this attenuation process. Notice that this proportion is independent of q. [*Note added in proof*. V. Ambegaokar and A. Griffin have independently derived Eq. (75) and numerically evaluated ρ^N/ρ. This work is reported in the Proceedings of the Ninth Low-Temperature Physics Conference (to be published). We would like to thank these authors for pointing out several errors in the preprint of our work.]

[19] A. B. Pippard, Phil. Mag. **46**, 1104 (1955).

[20] The extra factor of $g(ql)$ which appears in the transverse attenuation in Eq. (76) results from the greatly enhanced elimination of magnetic fields in the superconductor. L. T. Claiborne (Ref. 7) found this factor both experimentally and theoretically for the superconductor with nonmagnetic impurities. This type of effect was qualitatively predicted by R. W. Morse in *Progress in Cryogenics* (Heywood and Company, Ltd., London, 1959), Vol. 1, and Bardeen and Schrieffer, in *Progress in Low Temperature Physics*, edited by C. J. Gorter (North-Holland Publishing Company, Amsterdam, 1961), Vol. III, p. 170.

The integrations in (66) and (70) may be performed analytically in the limit of low temperatures. When $\Gamma_s/\Delta>1$, there is no gap in the energy spectrum, so that we can write $\partial f/\partial\omega=-\delta(\omega)$. At $\omega=0$,

$$\begin{aligned}c(\omega)&=2(\Delta/\Gamma_s)^2-1\\ y(\omega)&=ql\\ x(\omega)&=qv_F/[\Gamma_n+\Gamma_s-2\Delta^2/\Gamma_s]\,,\end{aligned} \tag{78}$$

so that

$$\rho^N/\rho=1-(\Delta/\Gamma_s)^2 \tag{79}$$

and

$$\begin{aligned}[\alpha_T/\alpha_T{}^N]/(\rho^N/\rho)&=g(ql)\\ &\to 1 \quad\text{for}\quad ql\ll1\,,\\ &\to 3\pi/4ql \quad\text{for}\quad ql\gg1\,,\end{aligned} \tag{80}$$

while

$$[\alpha_L/\alpha_L{}^N]/(\rho^N/\rho)$$
$$=\frac{[1-(ql)^{-1}\tan^{-1}(ql)][1+6\Delta^2/(\Gamma_s q^2v_F l)]}{1-[(ql)^{-1}-2\Delta^2/(\Gamma_s qv_F)]\tan^{-1}(ql)}$$
$$\to 1 \text{ for } ql\ll1 \text{ and for } ql\gg1\,. \tag{81}$$

On the other hand, when $(\Delta/\Gamma_s)>1$, there is a gap in the energy spectrum in the range $-\omega_g<\omega<\omega_g$, where

$$\omega_g=\Delta[1-(\Gamma_s/\Delta)^{2/3}]^{3/2}. \tag{82}$$

Just above the energy gap,

$$\begin{aligned}c(\omega)&=1-\tfrac{2}{3}(\omega-\omega_g)\Gamma_s{}^{-2/3}\omega_g{}^{-1/3}\,,\\ y(\omega)&=y_0=qv_F/[\Gamma_n-\Gamma_s+2\Delta(\Gamma_s/\Delta)^{1/3}]\,,\\ x(\omega)&=qv_F/[\Gamma_n-\Gamma_s]\,,\end{aligned} \tag{83}$$

so that

$$(\alpha_T/\alpha_T{}^N)/(\rho^N/\rho)=\frac{y_0{}^{-1}[1-g(y_0)]}{(ql)^{-1}[1-g(ql)]}g(ql)$$
$$\to\frac{\Gamma_n+\Gamma_s}{\Gamma_n-\Gamma_s+2\Delta(\Gamma_s/\Delta)^{1/3}} \quad\text{for } y_0\ll1\,,\quad ql\ll1\,,$$
$$\to\frac{3\pi}{4}\frac{\Gamma_n-\Gamma_s+2\Delta(\Gamma_s/\Delta)^{1/3}}{qv_F} \quad\text{for } ql\gg1\,. \tag{84}$$

Finally,

$$\begin{aligned}\frac{(\alpha_L/\alpha_L{}^N)}{(\rho^N/\rho)}&=\frac{\Gamma_n+\Gamma_s}{\Gamma_n-\Gamma_s+2\Delta(\Gamma_s/\Delta)^{1/3}} \quad\text{for } y_0\ll1 \text{ and } ql\ll1\,,\\ &=1 \quad\text{for } y_0\gg1\,.\end{aligned} \tag{85}$$

APPENDIX: DERIVATION OF FORMULA FOR ATTENUATION COEFFICIENT

Let[21] $\boldsymbol{\phi}(r,t)$ be the displacement for ions in the neighborhood of $\mathbf{r},t$. If the wavelength of the sound is much greater than the interatomic spacing, then $\boldsymbol{\phi}$ is perfectly well defined. Then, $\mathbf{F}=m\mathbf{a}$ for the ions is, for longitudinal fields,

$$M(\partial^2/\partial t^2)\boldsymbol{\phi}(\mathbf{r},t)=+Ze\mathbf{E}(\mathbf{r},t)+\mathbf{F}_{i,e}(\mathbf{r},t)\,, \quad \text{(A1)}$$

where $\mathbf{E}$ is the total electric field, Ze is the ionic charge, and $\mathbf{F}_{i,e}$ is any other force produced by the electrons which act on the ions.

For the electrons

$$\langle m(d\mathbf{j}/dt)(\mathbf{r},t)+\nabla\cdot\tau(\mathbf{r},t)\rangle=-Ne\mathbf{E}(\mathbf{r},t)+\mathbf{f}_{e,i}(\mathbf{r},t)\,, \quad \text{(A2)}$$

where $\mathbf{f}_{e,i}$ represents the residual force produced by the electrons on the ions. Newton's law, action=reaction implies that

$$(\mathbf{f}_{e,i}/N)Z=-\mathbf{F}_{i,e}\,, \quad \text{(A3)}$$

so that (A1) becomes

$$M\frac{\partial^2}{\partial t^2}\boldsymbol{\phi}(\mathbf{r},t)=-\frac{Z}{N}\left\langle\left[m\frac{d\mathbf{j}(\mathbf{r},t)}{dt}+\nabla\cdot\tau(\mathbf{r},t)\right]\right\rangle;$$

or, if $\boldsymbol{\phi}(\mathbf{r},t)$ varies as

$$\boldsymbol{\phi}(\mathbf{r},t)=\boldsymbol{\phi}(\mathbf{q},z)e^{i\mathbf{q}\cdot\mathbf{r}-izt}\,, \quad \text{(A4)}$$

(A1) becomes

$$-Mz^2\boldsymbol{\phi}(\mathbf{r},t)=-(Z/N)\langle[-imz\mathbf{j}(\mathbf{r},t)+\nabla\cdot\tau(\mathbf{r},t)]\rangle. \quad \text{(A5)}$$

Following Tsuneto,[6] we compute the right-hand side of (A5) by going into a coordinate system which moves with the ions. Then, in this system, we have an effective Hamiltonian for the electrons

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}_I\,, \quad \text{(A6)}$$

where $\mathcal{H}_0$ includes all electron-electron interactions including the Coulomb interaction. $\mathcal{H}_I$ is the effective Hamiltonian which is produced by the transition into the moving system:

$$\mathcal{H}_I=\int d^3r\boldsymbol{\phi}(r,t)\cdot[izm\mathbf{j}(\mathbf{r},t)-i\mathbf{q}\cdot\tau(\mathbf{r},t)]. \quad \text{(A7)}$$

Expression (A5) includes no electromagnetic interactions since these all appear in $\mathcal{H}_0$. No electromagnetic fields are directly produced by the transition into the moving system since this transition maintains average charge neutrality.

The transition into the moving system affects the right-hand side of (A5) in two ways: First there is the explicit change in the meaning of $\mathbf{j}$ and τ. From this explicit change, we find[6]

$$\begin{aligned}\langle -izmj_i(\mathbf{r},t)&+\nabla_j\tau_{ij}(\mathbf{r},t)\rangle_{\text{fixed}}\\ &=-z^2mN\phi_i(\mathbf{r},t)+q_jq_k\phi_k(\mathbf{r},t)\langle\tau_{ij}\rangle\\ &\quad+q_jq_j\phi_k(\mathbf{r},t)\langle\tau_{ik}\rangle+q_jq_i\phi_k(\mathbf{r},t)\langle\tau_{kj}\rangle\\ &\quad+\langle izmj_i(r,t)+\nabla_j\tau_{ij}(r,t)\rangle_{\text{moving}}\,, \quad \text{(A8)}\end{aligned}$$

where all repeated indices are to be taken as variables for summation.

The first four terms on the right-hand side of (A8) are already of first order in ϕ. Therefore, they can be calculated to zeroth order in $\mathcal{H}_I$. The last term in (A8) vanishes when $\mathcal{H}_I\to 0$. Therefore, it should be taken to first order in $\mathcal{H}_I$. Thus, we find that the right-hand side of (A8) is

$$\begin{aligned}-z^2mN\phi_i(\mathbf{r},t)&+\{2q_i[q\cdot\phi(\mathbf{r},t)]+q^2\phi_i(\mathbf{r},t)\}(Nmv_F^2/5)\\ &+\langle[h_i,h_j]\rangle(\mathbf{q},z)z^2\phi_j(\mathbf{r},t)\,, \quad \text{(A9)}\end{aligned}$$

where

$$h_i(\mathbf{r},t)=[q_j\tau_{ij}(\mathbf{r},t)/z]-mj_i(\mathbf{r},t). \quad \text{(A10)}$$

We substitute (A9) into the right-hand side of (A5), drop the first term in (A9) because it is of relative order Zm/M, and find

$$\begin{aligned}\rho_{\text{ion}}z^2\phi_i(\mathbf{q},z)&-\{2q_i[q\cdot\phi(\mathbf{q},z)]+q^2\phi_i(\mathbf{q},z)\}(mNv_F^2/5)\\ &+\langle[h_i,h_j]\rangle(\mathbf{q},z)z^2\phi_j(\mathbf{q},z)=0. \quad \text{(A11)}\end{aligned}$$

In the longitudinal case, ϕ_i is parallel to $\mathbf{q}$. We take these both to point in the z direction and then find, from (A11), the sound-wave dispersion relation

$$\rho_{\text{ion}}z^2=\tfrac{3}{5}Nmv_F^2q^2+\langle[h_I^L,h_I^L]\rangle(\mathbf{q},z)z^2\,, \quad \text{(A12)}$$

$$h_I^L(\mathbf{r},t)=(q/z)\tau_{zz}(\mathbf{r},t)-(mz/q)n(\mathbf{r},t). \quad \text{(A13)}$$

[In writing (A13), we applied the number conservation law $\partial n/\partial t+\nabla\cdot\mathbf{j}=0$ and dropped a term of relative order m/M.] From (A12), it follows at once that the longitudinal attenuation constant is

$$\alpha_L=-\text{Re}\frac{iz}{\rho_{\text{ion}}v_s}\langle[h_I^L,h_I^L]\rangle(\mathbf{q},z). \quad \text{(A14)}$$

The correlation functions in (A14) are honest electronic correlation functions including all effects of electron-electron Coulomb interactions. Now, we proceed to calculate in a more explicit fashion the effects of the Coulomb interactions.

With this aim in view, we notice that all our correlation functions may be viewed as variational derivatives, e.g.,

$$\langle[\tau_{ij},\tau_{kl}]\rangle(\mathbf{q},z)=[\delta\langle\tau_{ij}\rangle/\delta W_{kl}]_u(\mathbf{q},z)\,, \quad \text{(A15)}$$

where the expectation value is computed in a system with Coulomb interactions.

In order to state our problem in its most general form, we consider $\langle A(1)\rangle_{U,V}$ and $\langle A(1)\rangle'_{U,V}$, which are, respectively, expectation values of the same operator $A(\mathbf{r},t)$ in systems with and without Coulomb interac-

[21] The results of this Appendix grew out of many discussions that we have had with Dr. G. Baym. The calculational methods used here are very similar to ones which he derived, although our specific calculation is somewhat different.

tions. In each case, the expectation value is to be computed in the presence of a forcing term in the Hamiltonian

$$\delta\mathcal{H}=\int dr[U(\mathbf{r},t)n(\mathbf{r},t)+V(\mathbf{r},t)B(\mathbf{r},t)], \quad \text{(A16)}$$

where $V(\mathbf{r},t)$ is a c-number function of space and time and $B(\mathbf{r},t)$ is an operator.

In our calculations, we employ a generalization of the random phase approximation (R.P.A.),[22] in which we say that the total effect of introducing the long-ranged part of the Coulomb interaction can be taken into account by making the replacement

$$U(\mathbf{r},t)\to U_{\mathrm{eff}}(\mathbf{r},t) = U(\mathbf{r},t)+\int d^3r'\frac{e^2}{|\mathbf{r}-\mathbf{r}'|}n\langle(\mathbf{r}',t)\rangle \quad \text{(A17)}$$

wherever U appears in the theory without the long-range interactions. We use a prime to denote correlation functions in the fictitious system. Because of (A16)

$$\begin{aligned}\langle[A,B]\rangle'(\mathbf{q},z)&=\frac{\delta\langle A\rangle'}{\delta V}\bigg|_U(\mathbf{q},z),\\ \langle[A,n]\rangle'(\mathbf{q},z)&=\frac{\delta\langle A\rangle'}{\delta U}\bigg|_V(\mathbf{q},z),\\ \langle[n,n]\rangle'(\mathbf{q},z)&=\frac{\delta\langle n\rangle'}{\delta U}\bigg|_V(\mathbf{q},z),\\ \langle[n,B]\rangle'(\mathbf{q},z)&=\frac{\delta\langle n\rangle'}{\delta V}\bigg|_V(\mathbf{q},z).\end{aligned} \quad \text{(A18)}$$

Now consider the system with Coulomb interactions. Our assumption that the Coulomb interactions only appear through U_{eff} implies that

$$\frac{\delta\langle A\rangle}{\delta U_{\mathrm{eff}}}\bigg|_V(\mathbf{q},z)=\frac{\delta\langle A\rangle'}{\delta U}\bigg|_V(\mathbf{q},z)=\langle[A,n]\rangle'(\mathbf{q},z) \quad \text{(A19)}$$

and

$$\frac{\delta A}{\delta V}\bigg|_{U_{\mathrm{eff}}}(\mathbf{q},z)=\frac{\delta\langle A\rangle'}{\delta V}\bigg|_U=\langle[A,B]\rangle'(\mathbf{q},z). \quad \text{(A20)}$$

Also,

$$\frac{\delta\langle A\rangle}{\delta U}\bigg|_V(\mathbf{q},z)=\langle[A,n]\rangle(\mathbf{q},z)$$

is given by

$$\begin{aligned}\langle[A,n]\rangle(\mathbf{q},z)&=\frac{\delta\langle A\rangle}{\delta U_{\mathrm{eff}}}(\mathbf{q},z)\frac{\delta U_{\mathrm{eff}}}{\delta U}(\mathbf{q},z)\\ &=\frac{\delta A}{\delta U_{\mathrm{eff}}}(\mathbf{q},z)\left\{1+\frac{4\pi e^2}{q^2}\langle[n,n]\rangle(\mathbf{q},z)\right\}.\end{aligned} \quad \text{(A21)}$$

When $A=n$, (A21) reduces to

$$\langle[n,n]\rangle(\mathbf{q},z)=\frac{\langle[n,n]\rangle'(\mathbf{q},z)}{1-4\pi e^2q^{-2}\langle[n,n]\rangle'(\mathbf{q},z)}, \quad \text{(A22)}$$

which is a familiar R.P.A. result. By substituting (A22) into (A21), we find

$$\langle[A,n]\rangle(\mathbf{q},z)=\frac{\langle[A,n]\rangle'(\mathbf{q},z)}{1-4\pi e^2q^{-2}\langle[n,n]\rangle'(\mathbf{q},z)}. \quad \text{(A23)}$$

Now, consider

$$\begin{aligned}\langle[A,B]\rangle(\mathbf{q},z)&=\left[\frac{\delta\langle A\rangle}{\delta B}\right]_U(\mathbf{q},z)\\ &=\left[\frac{\delta\langle A\rangle}{\delta B}\right]_{U_{\mathrm{eff}}}(\mathbf{q},z)\\ &\quad+\left[\frac{\delta\langle A\rangle}{\delta U_{\mathrm{eff}}}\right]_V(\mathbf{q},z)\left[\frac{\delta U_{\mathrm{eff}}}{\delta V}\right](\mathbf{q},z)\\ &=\langle[A,B]\rangle'(\mathbf{q},z)\\ &\quad+\langle[A,n]\rangle'(\mathbf{q},z)\frac{4\pi e^2}{q^2}\langle[n,B]\rangle(\mathbf{q},z).\end{aligned} \quad \text{(A24)}$$

If we apply (A24) to the special case $A=n$, we find

$$\langle[n,B]\rangle(\mathbf{q},z)=\frac{\langle[n,B]\rangle'(\mathbf{q},z)}{1-4\pi e^2q^{-2}\langle[n,n]\rangle'(\mathbf{q},z)}, \quad \text{(A25)}$$

so that (A24) becomes

$$\langle[A,B]\rangle(\mathbf{q},z)=\langle[A,B]\rangle'(\mathbf{q},z)+\frac{\langle[A,n]\rangle'(\mathbf{q},z)4\pi e^2q^{-2}\langle[n,B]\rangle'(\mathbf{q},z)}{1-4\pi e^2q^{-2}\langle[n,n]\rangle'(\mathbf{q},z)}. \quad \text{(A26)}$$

To derive Eq. (3) of the text, we substitute (A26) into (A14).[23] In the text, we have implicitly assumed that all correlation functions are to be evaluated in the system without long-range correlations. Hence all primes are left out in the text.

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[22] P. Nozières and D. Pines, Nuovo Cimento 9, 470 (1958).

[23] We also use $\langle[n,\tau_{ij}]\rangle(\mathbf{q},z)=\langle[\tau_{ij},n]\rangle(\mathbf{q},z)$, which is derivable from parity and time-reversal invariance. See L. P. Kadanoff and P. C. Martin, Ann. Phys. (N. Y.) 24, 419 (1963).